

Precalculus · Solving Systems with Matrices · Lesson

Common Core A-REI.C.8, A-REI.C.9, N-VM.C.11, N-VM.C.12 · mathovore.fr

This precalculus matrix systems lesson explains solving systems with matrices in one printable page, with definitions, a method and a worked example.

A matrix equation $AX = B$ packs a linear system into one statement, where A holds the coefficients, X the unknowns and B the constants. When $\det A \neq 0$, the inverse gives the unique solution $X = A^{-1}B$. The same matrices also transform the plane, and $|\det A|$ is the factor by which areas change.

Precalculus Common Core: A-REI.C.8 · A-REI.C.9 · N-VM.C.11 · N-VM.C.12

Precalculus Matrix Systems Lesson: key facts

- A system of linear equations can be written as $AX = B$.
- For a 2×2 matrix, $\det A = ad - bc$ and $A^{-1} = \frac{1}{ad - bc}$ times the matrix with rows d , $-b$ and $-c$, a .
- If $\det A \neq 0$, then $X = A^{-1}B$ is the only solution; if $\det A = 0$, there is no inverse.
- The columns of a 2×2 matrix are the images of $(1, 0)$ and $(0, 1)$.
- A transformation by M multiplies every area by $|\det M|$.

How do you write a system as a matrix equation?

A system of linear equations packs into one equation, $AX = B$. The **coefficient matrix** A lists the coefficients, the column X holds the unknowns, and the column B holds the constants. The system $2x + 3y = 8$, $x - y = -1$ becomes the product below.

$$2x + 3y = 8 \text{ and } x - y = -1$$

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

A coefficients X unknowns B constants

$$AX = B$$

Each row of A times the column X reproduces one equation.

How do you find the inverse of a 2 by 2 matrix?

Inverse of a 2 by 2 matrix

For a matrix with first row a, b and second row c, d , the **determinant** is $ad - bc$. When it is not zero, the inverse has first row $d, -b$ and second row $-c, a$, all multiplied by $\frac{1}{ad - bc}$. Then $AA^{-1} = A^{-1}A = I$, where I is the identity matrix with rows 1, 0 and 0, 1.

If $ad - bc = 0$, the matrix is **singular** and has no inverse.

General rule

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\text{swap}} \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

swap a and d, negate b and c

Example

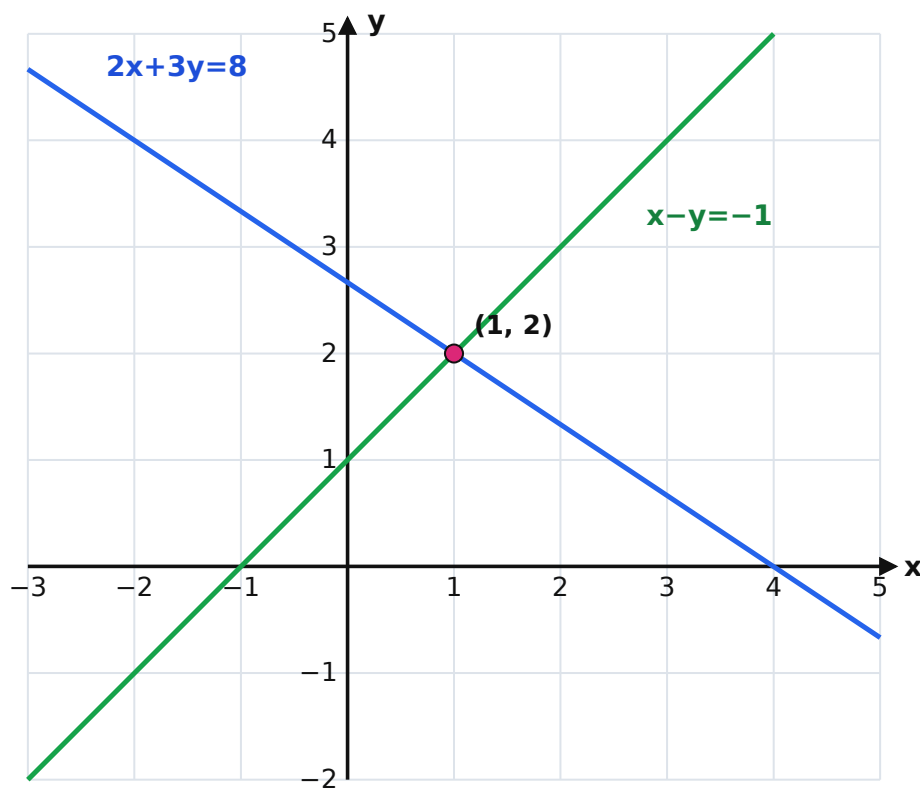
$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \rightarrow \frac{1}{-5} \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1/5 & 3/5 \\ 1/5 & -2/5 \end{bmatrix}$$

Swap the diagonal, negate the other two entries, divide by the determinant.

How do you solve $AX = B$ with the inverse?

1. Write the system as $AX = B$.
2. Compute $\det A$. If it is 0, stop: there is no unique solution.
3. Build A^{-1} .
4. Multiply on the left: $X = A^{-1}B$. Order matters, because matrix multiplication is not commutative.
5. Check the answer in both original equations.

Example. For A above, $\det A = 2(-1) - 3(1) = -5$. So A^{-1} has rows $\frac{1}{5}, \frac{3}{5}$ and $\frac{1}{5}, -\frac{2}{5}$. Then $x = \frac{1}{5}(8) + \frac{3}{5}(-1) = 1$ and $y = \frac{1}{5}(8) - \frac{2}{5}(-1) = 2$. Check: $2(1) + 3(2) = 8$ and $1 - 2 = -1$. The solution is $(1, 2)$.

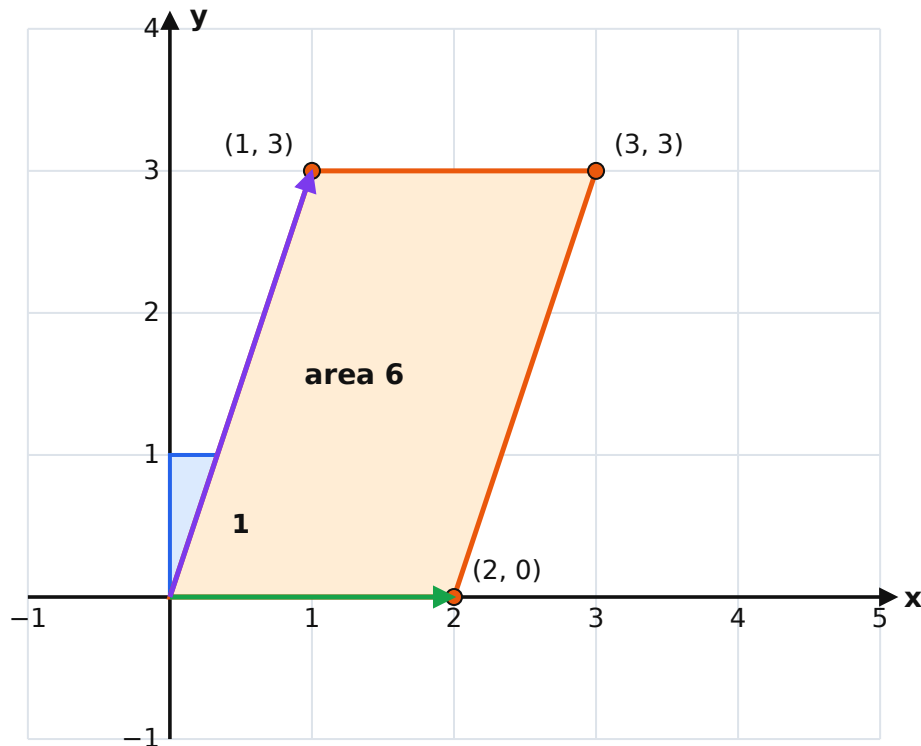


The solution $(1, 2)$ is where the two lines cross.

For a 3×3 or larger system, a graphing calculator or software finds A^{-1} ; the method $X = A^{-1}B$ stays the same.

How does a matrix transform the plane?

Multiplying a point (x, y) , written as a column, by a 2×2 matrix M moves it to a new point. The two columns of M are the images of $(1, 0)$ and $(0, 1)$. For M with columns $(2, 0)$ and $(1, 3)$, the unit square lands on the parallelogram below. For example, $(1, 1)$ goes to $(2+1, 0+3) = (3, 3)$.

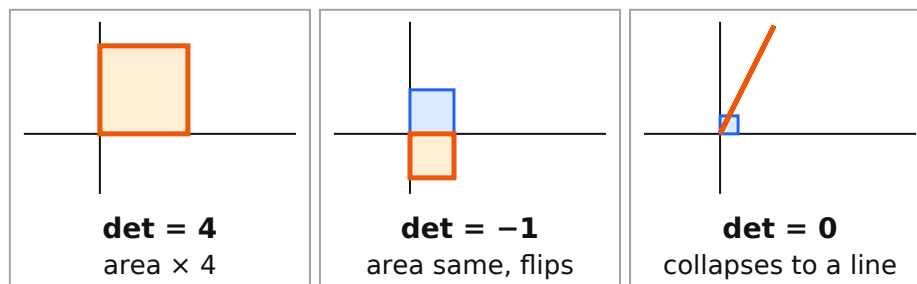


Green and violet arrows: the images of $(1, 0)$ and $(0, 1)$.

What does the determinant say about area?

The image of any region has area $|\det M|$ times the original area. Here $\det M = 2 \cdot 3 - 1 \cdot 0 = 6$, so the unit square (area 1) becomes a parallelogram of area 6. A negative determinant flips orientation without changing the area rule. A determinant of 0 squashes the plane onto a line, so no inverse can undo it.

Image of the unit square under three matrices



Determinant 4, -1 and 0 describe three very different behaviors.

Solving Systems with Matrices: frequently asked questions

What is the inverse of a matrix?

The inverse A^{-1} is the matrix that undoes A , so that $AA^{-1} = A^{-1}A = I$. Only square matrices with a nonzero determinant have one.

Why does the order in $X = A^{-1}B$ matter?

Matrix multiplication is not commutative. To cancel A from $AX = B$, multiply on the left of both sides by A^{-1} , which gives $X = A^{-1}B$, not BA^{-1} .

What happens when the determinant is zero?

The matrix has no inverse. The lines of a 2 by 2 system are then parallel (no solution) or the same line (infinitely many solutions).

How do you solve a 3 by 3 system with matrices?

Write it as $AX = B$, enter A and B into a graphing calculator or software, compute A^{-1} , and multiply $A^{-1}B$. By hand the method is long, so technology is used.

What does the determinant mean geometrically?

For a 2×2 matrix, $|\det|$ is the area of the image of the unit square, and so the area scale factor of the whole transformation. A negative sign means the plane is flipped.

How to use this precalculus matrix systems lesson

Use this precalculus matrix systems lesson before the worksheet, then check what you learned with the quiz.

