

Precalculus · Complex Factorization and Volume of a Sphere · Lesson

Common Core N-CN.C.8, G-GMD.A.2 · mathovore.fr

This precalculus complex identities volume lesson explains complex factorization and volume of a sphere in one printable page, with definitions, a method and a worked example.

Over the complex numbers, a sum of squares factors: a squared plus b squared equals $(a + bi)(a - bi)$, so $x^2 + 4$ equals $(x + 2i)(x - 2i)$. Cavalieri's principle shows that solids with equal cross-sectional areas have equal volume, which gives the sphere volume formula $\frac{4}{3} \pi r^3$.

Precalculus Common Core: N-CN.C.8 · G-GMD.A.2

Precalculus Complex Identities Volume Lesson: key facts

- $a^2 + b^2 = (a + bi)(a - bi)$; for example $x^2 + 4 = (x + 2i)(x - 2i)$.
- Familiar polynomial identities remain true when the variables are complex numbers.
- Cavalieri's principle: equal cross-sectional areas at every height mean equal volumes.
- A hemisphere matches a cylinder minus a cone, so its volume is $\frac{2}{3} \pi r^3$.
- The volume of a sphere is $V = \frac{4}{3} \pi r^3$.

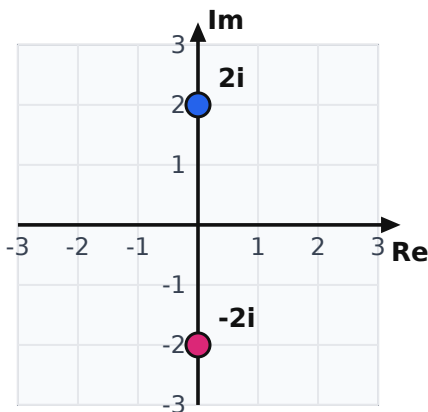
How do you factor a sum of squares over the complex numbers?

The polynomial $x^2 + 4$ has no real factors, because a real square is never negative. Over the **complex numbers** it does factor. Since $i^2 = -1$, the product $(x + 2i)(x - 2i) = x^2 - (2i)^2 = x^2 - 4i^2 = x^2 + 4$. In general, a sum of squares is a difference of squares in disguise:

Sum of squares identity

$a^2 + b^2 = (a + bi)(a - bi)$ for any numbers a and b . The two factors are complex conjugates.

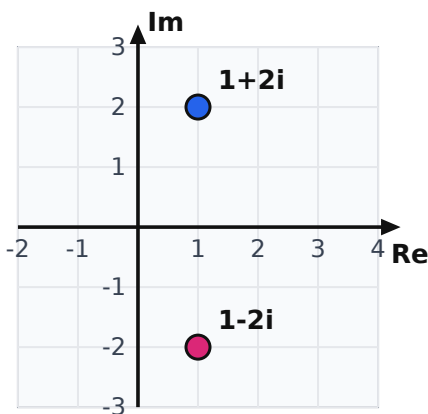
The identities you know from algebra, such as the difference of squares and $(a + b)^2 = a^2 + 2ab + b^2$, stay true when the letters stand for complex numbers. That is why the same factoring methods keep working.



The zeros of x^2+4 are the points $2i$ and $-2i$.

1. Write the polynomial as a square plus a square: $9x^2+25=(3x)^2+5^2$.
2. Apply $a^2+b^2=(a+bi)(a-bi)$ with $a=3x$ and $b=5$.
3. Result: $9x^2+25=(3x+5i)(3x-5i)$. The zeros are $x=\pm\frac{5}{3}i$.

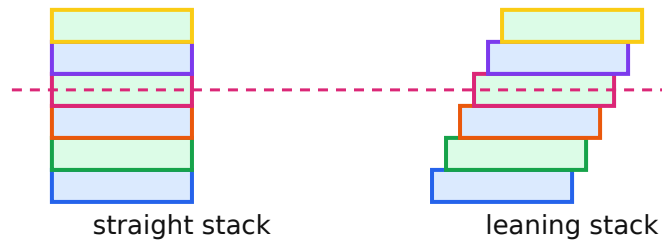
For a quadratic such as x^2-2x+5 , complete the square first: $(x-1)^2+4=(x-1+2i)(x-1-2i)$, so the zeros are $1\pm 2i$.



Zeros of x^2-2x+5 : a conjugate pair.

What is Cavalieri's principle?

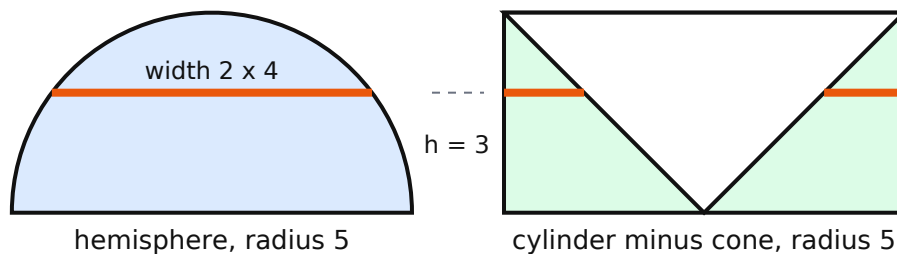
Cavalieri's principle says that if two solids lie between the same two parallel planes, and every plane parallel to them cuts both solids in cross-sections of equal area, then the solids have equal volume. A stack of coins shows the idea: sliding the coins sideways into a leaning stack changes the shape but not the volume.



Every level cuts both stacks in the same circle, so the volumes match.

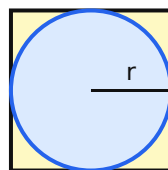
How does Cavalieri's principle give the volume of a sphere?

Compare a hemisphere of radius r with a cylinder of radius r and height r from which a cone has been removed (apex at the center of the bottom, base on top). Cut both at height h . The hemisphere gives a circle of radius $\sqrt{r^2 - h^2}$ and area $\pi(r^2 - h^2)$. The cylinder with the cone removed gives a ring with outer radius r and inner radius h , area $\pi r^2 - \pi h^2$. The areas match at every height.



At height 3 with $r = 5$, both cross-sections have area 16π .

So the volumes are equal: hemisphere $= \pi r^2 \cdot r - \frac{1}{3} \pi r^2 \cdot r = \frac{2}{3} \pi r^3$. Doubling gives the sphere: $V = \frac{4}{3} \pi r^3$. For $r = 3$, $V = \frac{4}{3} \pi (27) = 36\pi \approx 113.1$ cubic units.



A sphere inside a snug cylinder: its volume is two thirds of the cylinder's.

Complex Factorization and Volume of a Sphere: frequently asked

questions

Why can $x^2 + 4$ be factored with complex numbers but not with real numbers?

Its zeros are $2i$ and $-2i$, which are not real. Over the complex numbers, each zero gives a linear factor, so the polynomial becomes $(x + 2i)(x - 2i)$.

Does the sum of squares identity work for any a and b ?

Yes. Expanding $(a + bi)(a - bi)$ gives $a^2 - b^2 i^2$, which equals $a^2 + b^2$. It works for real or complex a and b .

What does Cavalieri's principle require?

Both solids must sit between the same two parallel planes, and every plane parallel to those planes must cut the two solids in regions of equal area.

Why is a cone removed from the cylinder when finding the volume of a sphere?

At height h , the cone's cross-section has radius h , so the leftover ring has area $\pi r^2 - \pi h^2$. That equals the hemisphere's cross-section at the same height.

How do you find the volume of a sphere with a given diameter?

Divide the diameter by 2 to get the radius, then use $V = \frac{4}{3} \pi r^3$. A sphere with diameter 10 has radius 5 and volume $\frac{500\pi}{3}$, about 523.6 cubic units.

How to use this precalculus complex identities volume lesson

Use this precalculus complex identities volume lesson before the worksheet, then check what you learned with the quiz.

