

Geometry · Area and Volume Formulas · Lesson

Common Core G-GMD.A.1, G-GMD.A.3 · mathovore.fr

The circumference of a circle is $C = 2\pi r$ and its area is $A = \pi r^2$. Cutting a circle into sectors and rearranging them explains the area formula. Cavalieri's principle then gives the volumes of a cylinder, $\pi r^2 h$, a pyramid, $(1/3)Bh$, a cone, $(1/3)\pi r^2 h$, and a sphere, $(4/3)\pi r^3$.

Geometry Common Core: G-GMD.A.1 · G-GMD.A.3

Key facts:

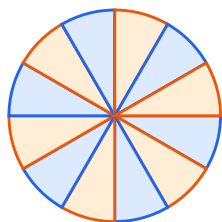
- Standards G-GMD.A.1 and G-GMD.A.3 cover informal arguments for circle and solid formulas, and solving volume problems.
- Circumference $C = 2\pi r$ and area $A = \pi r^2$ both come from the constant $\pi = C/d$.
- Cavalieri's principle: equal heights and equal cross-sectional areas at every level mean equal volumes.
- Cone and pyramid volumes are one third of the matching cylinder or prism: $V = (1/3)Bh$.
- Sphere volume $V = (4/3)\pi r^3$ equals two hemispheres, each a cylinder minus a cone.

Why is the circumference of a circle $2\pi r$?

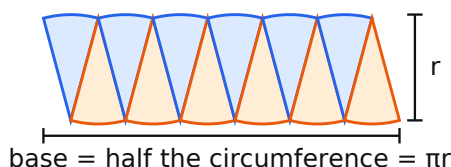
Every circle is a scaled copy of every other circle, so the ratio of circumference to diameter is the same number for all of them. That number is π (about 3.14159). From $\frac{C}{d} = \pi$ we get $C = \pi d = 2\pi r$.

How can you see that the area of a circle is πr^2 ?

Cut a circle into thin sectors and lay them side by side, point up then point down. The more sectors you cut, the closer the shape gets to a parallelogram. Its base is half the circumference, πr , and its height is the radius r . So $A = \pi r \cdot r = \pi r^2$.



Circle cut into 12 sectors



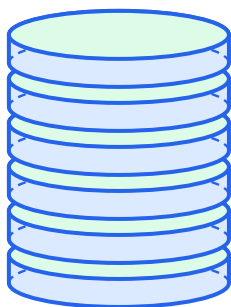
Rearranged sectors form a shape close to a parallelogram.

What is Cavalieri's principle?

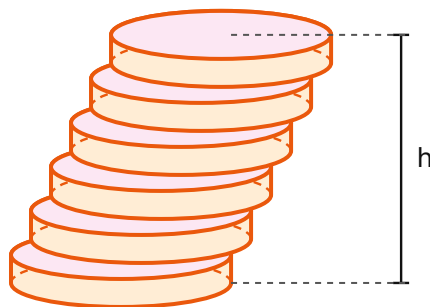
Cavalieri's principle

If two solids have the same height and the same cross-sectional area at every level, then they have the same volume.

A stack of coins keeps its volume when you push it into a slant, because each coin still has the same area. A slanted cylinder therefore has the same volume as a right cylinder with the same base and height.



Straight stack



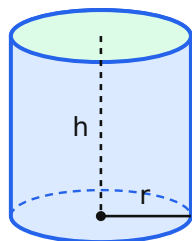
Slanted stack

Same coins, same height: same volume.

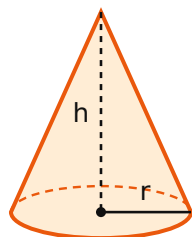
What are the volume formulas for cylinders, pyramids, cones, and spheres?

Solid	Volume	Why it works
Cylinder	$V = \pi r^2 h$	Cavalieri: same base area and height as a prism, so $V = Bh$

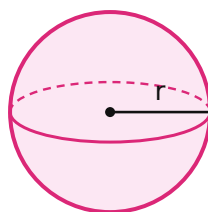
Solid	Volume	Why it works
Pyramid	$V = \frac{1}{3}Bh$	Dissection: three congruent pyramids fill a prism
Cone	$V = \frac{1}{3}\pi r^2 h$	Cavalieri: matches a pyramid with equal base area and height
Sphere	$V = \frac{4}{3}\pi r^3$	Cavalieri: hemisphere = cylinder minus cone



Cylinder
 $V = \pi r^2 h$



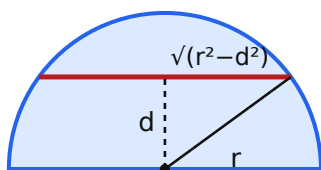
Cone
 $V = (1/3)\pi r^2 h$



Sphere
 $V = (4/3)\pi r^3$

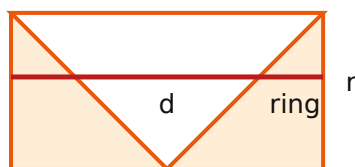
The three round solids and their volume formulas.

For the sphere, a slice of a hemisphere at height d is a circle of area $\pi(r^2 - d^2)$. A cylinder with a cone removed has a ring-shaped slice of the same area at every height. So the hemisphere has volume $\pi r^3 - \frac{1}{3}\pi r^3 = \frac{2}{3}\pi r^3$.



Hemisphere

=



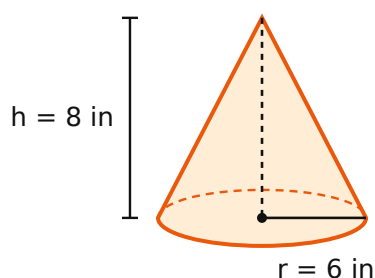
Cylinder minus cone

At every height d , the hemisphere slice and the ring have equal area: $\pi(r^2 - d^2)$.

How do you use the formulas to solve a problem?

1. Name the solid and write its formula.
2. Convert a diameter to a radius.
3. Substitute the measurements and keep π in the exact answer.
4. Approximate with a calculator and attach cubic units.

Worked example. A cone has radius 6 in and height 8 in. Then $V = \frac{1}{3}\pi(6)^2(8) = 96\pi \approx 301.59 \text{ in}^3$.



The cone from the worked example.

Area and Volume Formulas: frequently asked questions

Why is the ratio of circumference to diameter the same for every circle?

Any two circles are similar, meaning one is an enlarged copy of the other. Enlarging multiplies both the circumference and the diameter by the same factor, so their ratio never changes. That constant ratio is π .

What does Cavalieri's principle say in simple words?

Slice two solids into layers at the same heights. If every pair of matching layers has the same area, the solids have the same volume, even if one is slanted.

Why does the cone formula have a factor of one third?

Three congruent pyramids fit together to fill a prism with the same base and height. A cone matches a pyramid with equal base area and height, so its volume is one third of the cylinder with the same base and height.

Do the volume formulas use the radius or the diameter?

The formulas use the radius. If a problem gives a diameter, cut it in half first. Using the diameter by mistake makes a cylinder volume four times too big.

What units does volume have?

Volume is measured in cubic units, such as cm^3 , in^3 or ft^3 , because it counts unit cubes that fill the solid. Area uses square units and length uses plain units.