

Geometry · Probability: Independence, Conditional Probability and Fair Decisions · Lesson

Common Core S-CP.A.1, S-CP.A.2, S-CP.A.3, S-CP.A.4, S-CP.A.5, S-CP.B.6, S-CP.B.7, S-CP.B.8, S-CP.B.9, S-MD.B.6, S-MD.B.7 · mathovore.fr

Probability describes events on a sample space using and, or and not. Conditional probability, $P(A \text{ given } B) = P(A \text{ and } B) / P(B)$, measures the chance of A once B is known, and events are independent when $P(A \text{ and } B) = P(A)P(B)$. Addition and multiplication rules, counting, and natural frequencies support fair random choices and decisions such as medical tests.

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Key facts:

- Addition rule: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$; complement: $P(\text{not } A) = 1 - P(A)$.
- Independent events satisfy $P(A \text{ and } B) = P(A)P(B)$, which is the same as $P(A \text{ given } B) = P(A)$.
- Conditional probability: $P(A \text{ given } B) = P(A \text{ and } B) / P(B)$; general multiplication rule: $P(A \text{ and } B) = P(B)P(A \text{ given } B)$.
- Permutations count ordered selections, $nPr = n!/(n-r)!$; combinations count unordered ones, $nCr = n!/(r!(n-r)!)$.
- A random decision is fair when every choice has the same probability; natural frequencies make medical-test probabilities clear.

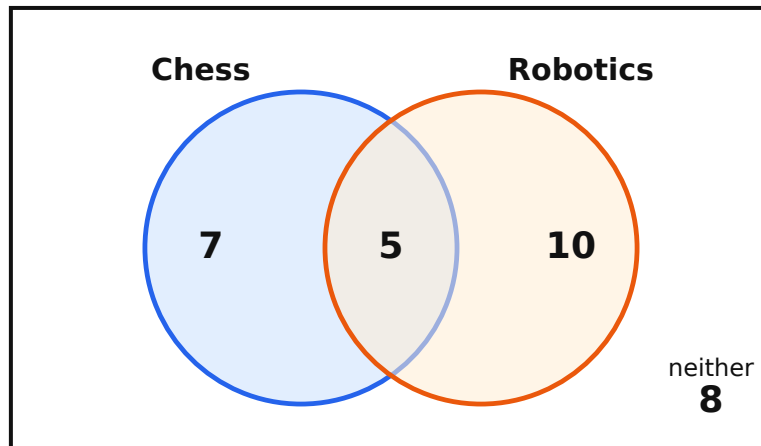
What are a sample space, a union, an intersection and a complement?

The **sample space** is the set of all possible outcomes of a random process, and an **event** is any part of it. Events are built with the words *and*, *or* and *not*.

Event	Meaning	Probability
A and B	Outcomes in both events	$P(A \text{ and } B)$
A or B	In A, in B, or in both	$P(A)+P(B)-P(A \text{ and } B)$

Event	Meaning	Probability
not A	Everything outside A	$1 - P(A)$

Example: of 30 students, 12 play chess, 15 do robotics and 5 do both, so $P(\text{chess or robotics}) = \frac{12+15-5}{30} = \frac{11}{15}$ and $P(\text{neither}) = \frac{8}{30} = \frac{4}{15}$.



30 students: 12 play chess, 15 do robotics, 5 do both. The overlap is counted once.

What does it mean for two events to be independent?

Independent events

Events A and B are independent when $P(A \text{ and } B) = P(A) \cdot P(B)$. Equivalently, $P(A|B) = P(A)$: learning that B happened does not change the chance of A.

How do you find conditional probability from a two-way table?

The conditional probability of A given B is $P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$. In a table, restrict attention to the row or column for B. Rearranged, it gives the general multiplication rule $P(A \text{ and } B) = P(B) \cdot P(A|B)$.

In the table below, $P(\text{job}|\text{10th}) = \frac{30}{60} = \frac{1}{2}$, while $P(\text{job}|\text{9th}) = \frac{18}{60} = \frac{3}{10}$. The two answers differ, so having a job is not independent of grade.

Grade \ Job	Has job	No job	Total
9th grade	18	42	60
10th grade	30	30	60
Total	48	72	120

The highlighted cell holds the 30 tenth graders with a job.

How do permutations and combinations count outcomes?

When order matters, count **permutations**: ${}_nP_r = \frac{n!}{(n-r)!}$. When order does not matter, count **combinations**: ${}_nC_r = \frac{n!}{r!(n-r)!}$. Choosing 3 of 10 students gives ${}_{10}C_3 = 120$ committees, and each one is equally likely.

How do probabilities support fair decisions and strategies?

A random method is **fair** when every person has the same probability of being chosen. Maya, Jamal and Sofia can roll a die and give each friend two faces.

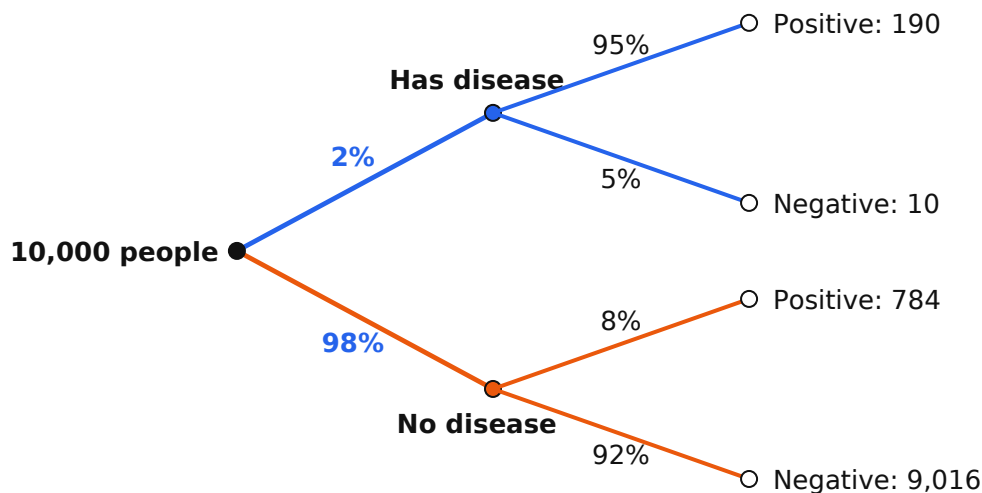
1	2	3	4	5	6
Maya	Maya	Jamal	Jamal	Sofia	Sofia

Die result (top) and who is chosen (bottom)

Two faces per person: each friend is chosen with probability $2/6 = 1/3$.

Probabilities also test strategies. To judge a medical test, use natural frequencies:

1. Start with a population of 10,000 people.
2. Split it by condition: 2% have the disease, so 200 do and 9,800 do not.
3. Apply the test rates: 95% of 200 is 190 true positives; 8% of 9,800 is 784 false positives.
4. Divide: $P(\text{disease}|+) = \frac{190}{190+784} \approx 0.195$.



Of 974 positive tests (190 + 784), only 190 are true positives.

Even with a positive test, the chance of having the disease is only about 19.5%, because the disease is rare.

Probability: Independence, Conditional Probability and Fair Decisions: frequently asked questions

What is the difference between independent and mutually exclusive events?

Mutually exclusive events cannot happen together, so $P(A \text{ and } B) = 0$. Independent events can happen together, and one does not change the chance of the other. Two events with positive probabilities can never be both.

How do you check whether two events are independent?

Compute $P(A)$ times $P(B)$ and compare it with $P(A \text{ and } B)$. If they are equal, the events are independent. In a two-way table, you can also compare $P(A \text{ given } B)$ with $P(A)$.

Why is $P(A \text{ given } B)$ usually not equal to $P(B \text{ given } A)$?

The two probabilities use different totals. $P(A \text{ given } B)$ divides by $P(B)$, while $P(B \text{ given } A)$ divides by $P(A)$. In a table they use different row or column totals.

When do you use a combination instead of a permutation?

Use a combination when the order of selection does not matter, such as picking a committee or a lottery ticket. Use a permutation when order matters, such as awarding gold, silver and bronze medals.

How can a positive medical test still mean a low chance of disease?

If the disease is rare, most people tested are healthy, so even a small false positive rate produces many false positives. With a 2% rate, false positives can outnumber true positives four to one.