

Geometry · Parabolas from Focus and Directrix, and Tangent Lines to a Circle · Lesson

Common Core G-GPE.A.2, G-C.A.4 · mathovore.fr

A parabola is the set of points equally far from a focus and a directrix. With vertex (h, k) , its equation is $(x - h)^2 = 4p(y - k)$, where p is the vertex-to-focus distance. A tangent to a circle is perpendicular to the radius at the point of tangency, so tangent segments from one point are equal.

GeometryCommon Core: G-GPE.A.2 · G-C.A.4

Key facts:

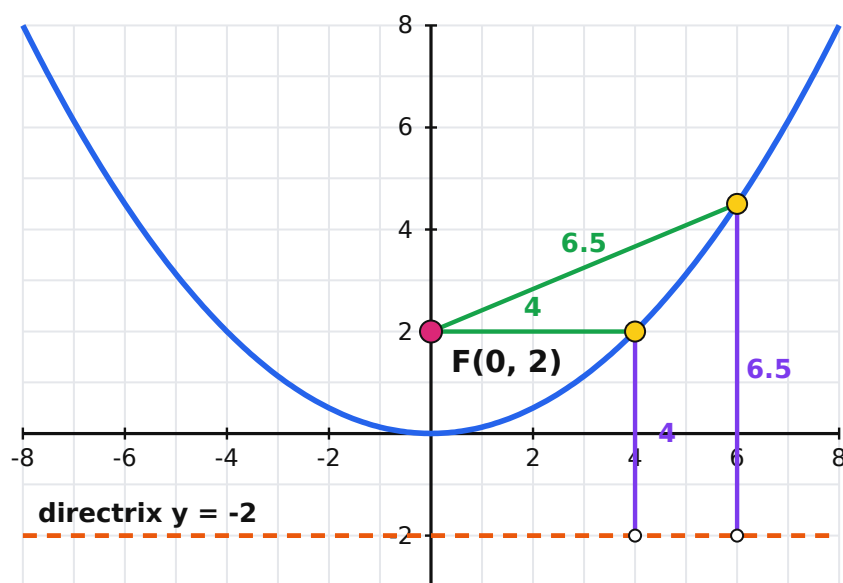
- A parabola is the set of points equidistant from the focus and the directrix (G-GPE.A.2).
- Vertex (h, k) : $(x - h)^2 = 4p(y - k)$ has focus $(h, k + p)$ and directrix $y = k - p$; $(y - k)^2 = 4p(x - h)$ opens sideways.
- p is the signed distance from the vertex to the focus; a negative p means the parabola opens down or left.
- A tangent to a circle is perpendicular to the radius at the point of tangency (G-C.A.4).
- Tangent segments from one outside point are congruent; the circle on diameter OP gives the tangent points in a construction.

What is a parabola?

Parabola

A **parabola** is the set of all points that are the same distance from a fixed point, the **focus**, and a fixed line, the **directrix**. The point halfway between them on the curve is the **vertex**.

For the focus $F(0,2)$ and directrix $y = -2$, the point $(4,2)$ is 4 units from F and 4 units above the directrix, so it is on the curve.



Every point on the curve is as far from the focus F as from the directrix.

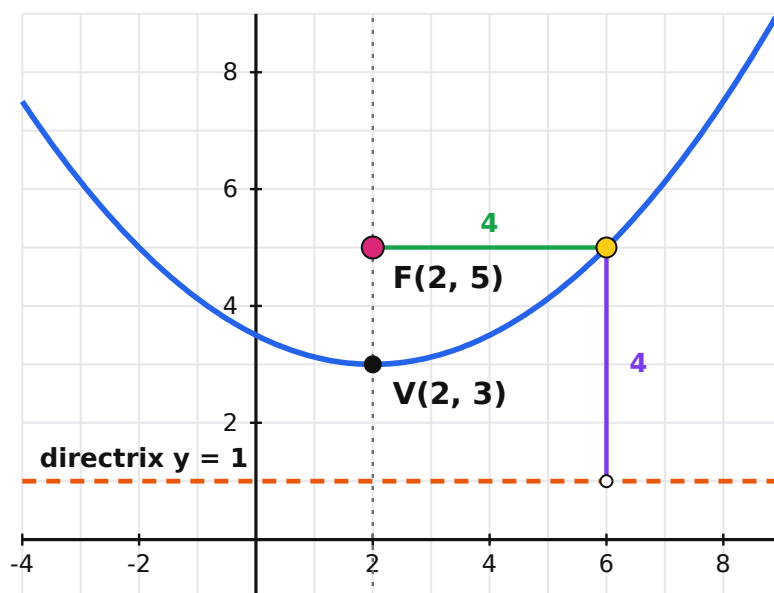
How do you write the equation from a focus and a directrix?

1. Find the vertex, the midpoint between the focus and the directrix, and the signed distance p from the vertex to the focus.
2. Choose the form that matches the opening direction.
3. Substitute the vertex (h, k) and p .

Opens	Equation	Focus	Directrix
Up or down	$(x-h)^2 = 4p(y-k)$	$(h, k+p)$	$y = k-p$
Right or left	$(y-k)^2 = 4p(x-h)$	$(h+p, k)$	$x = h-p$

The distance formula explains the first row for a vertex at the origin: $\sqrt{x^2 + (y-p)^2} = |y+p|$ squares to $x^2 = 4py$.

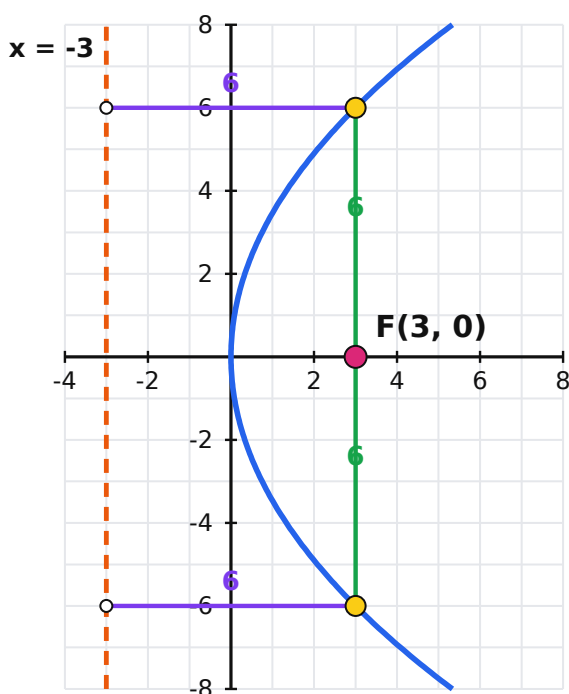
Worked example. The focus is $(2, 5)$ and the directrix is $y = 1$. The vertex is $(2, 3)$ and $p = 2$, so $(x-2)^2 = 8(y-3)$.



Vertex $V(2, 3)$, $p = 2$: halfway between focus and directrix.

How do you find the focus and directrix from an equation?

Match the equation to a form and read off $4p$. For $y^2 = 12x$, $4p = 12$ gives $p = 3$, so the focus is $(3, 0)$ and the directrix is $x = -3$. A negative p means the parabola opens down or left.



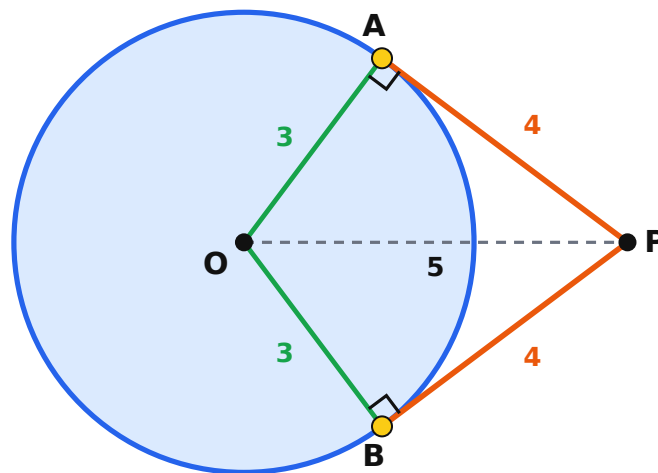
A parabola that opens to the right: $y^2 = 12x$, $p = 3$.

What is special about a tangent to a circle?

Tangent line

A **tangent** touches a circle at exactly one point and is perpendicular to the radius drawn to that point. Two tangent segments from the same outside point are congruent.

With radius 3 and $OP = 5$, the right triangle gives $PA = \sqrt{5^2 - 3^2} = 4$.

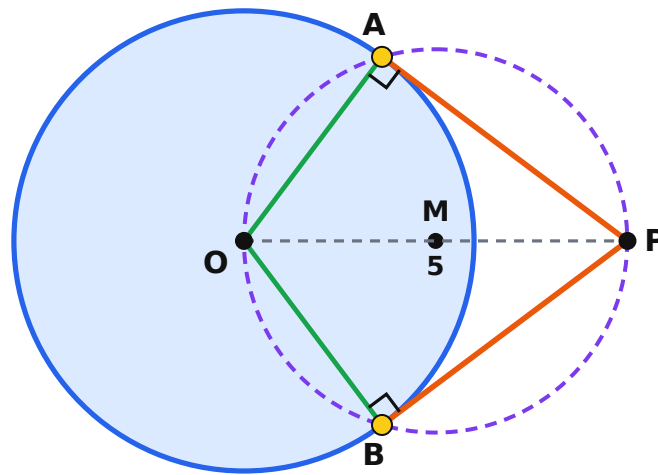


Tangent segments from P: $PA = PB = 4$, and each is perpendicular to its radius.

How do you construct a tangent from an outside point?

1. Draw segment OP from the center to the point.
2. Construct the midpoint M of OP .
3. Draw the circle with center M and radius MO .
4. Mark the points A and B where it meets the given circle.
5. Draw lines PA and PB .

Why it works: $\angle OAP$ is inscribed in a semicircle with diameter OP , so it is a right angle and PA is perpendicular to the radius OA .



Steps 3 and 4: the dashed circle centered at M meets the given circle at A and B.

Parabolas from Focus and Directrix, and Tangent Lines to a Circle: frequently asked questions

What is the focus of a parabola?

The focus is the fixed point used in the definition of a parabola. Every point of the curve is as far from the focus as from the directrix. The focus lies inside the curve on its axis of symmetry.

How do you find p in a parabola equation?

Write the equation as $(x - h)^2 = 4p(y - k)$ or $(y - k)^2 = 4p(x - h)$ and divide the coefficient by 4. For $x^2 = 12y$, $4p = 12$ and $p = 3$.

Why is a tangent perpendicular to the radius?

The radius to the point of tangency is the shortest path from the center to the tangent line, because every other point of the line lies outside the circle. The shortest path to a line is the perpendicular.

Why are two tangent segments from the same point equal?

Right triangles OAP and OBP share the hypotenuse OP and have equal legs OA and OB, the radii. The Hypotenuse-Leg theorem makes them congruent, so $PA = PB$.

How can the tangent length be found without a construction?

Use the Pythagorean theorem in right triangle OAP. If the radius is r and the distance from the center to the point is d , the tangent length is the square root of $d^2 - r^2$.