

AP Calculus AB · Applications of Integration · Worksheet Answer Key

Common Core AP Calculus AB Unit 8 · mathovore.fr

Every worked solution for the applications of integration worksheet is below. Each shows the setup, the antiderivative and the evaluation, with the final answer in bold. Compare your setup first, because most lost points come from the integrand or the limits.

AP Calculus ABCommon Core: AP Calculus AB Unit 8

Problem 1: Average value of a polynomial

$$\int_1^4 (3x^2 - 2x) dx = [x^3 - x^2]_1^4 = 48 - 0 = 48.$$

$$g_{\text{avg}} = \frac{1}{4-1} \cdot 48 = 16.$$

16

Problem 2: Average value of sine

$$\int_0^\pi \sin x dx = [-\cos x]_0^\pi = 1 - (-1) = 2.$$

$$f_{\text{avg}} = \frac{2}{\pi} \approx 0.6366.$$

$$\frac{2}{\pi} \approx 0.6366$$

Problem 3: Average value of a quadratic

An antiderivative is $x^3 - x^2 + x$. At 3: $27 - 9 + 3 = 21$. At 1: $1 - 1 + 1 = 1$. The integral is 20.

$$f_{\text{avg}} = \frac{20}{2} = 10.$$

10

Problem 4: Where the average value occurs

The average value is 3 (from the first problem). Solve $c^2 = 3$: $c = \pm\sqrt{3}$. Only $\sqrt{3}$ is in $(0,3)$.

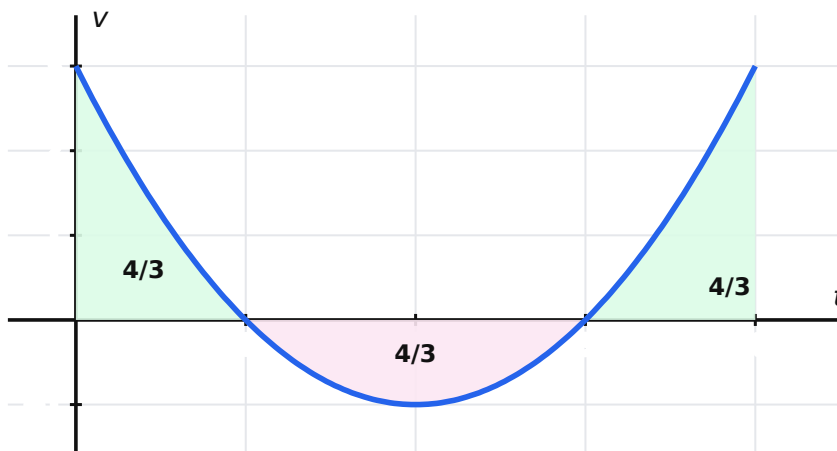
$$c = \sqrt{3} \approx 1.732$$

Problem 5: Displacement and distance

$v(t) = (t-1)(t-3)$, so the sign changes at $t=1$ and $t=3$. An antiderivative is $\frac{t^3}{3} - 2t^2 + 3t$.

$$\int_0^1 v \, dt = \frac{4}{3}, \quad \int_1^3 v \, dt = -\frac{4}{3}, \quad \int_3^4 v \, dt = \frac{4}{3}.$$

Displacement: $\frac{4}{3} - \frac{4}{3} + \frac{4}{3} = \frac{4}{3}$. Distance: $\frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4$.



Each shaded region has area $\frac{4}{3}$; the green ones are above the axis.

Displacement $\frac{4}{3}$; total distance 4

Problem 6: Final position

$$\int_0^2 (3t^2 - 6t) \, dt = [t^3 - 3t^2]_0^2 = 8 - 12 = -4.$$

$$s(2) = 2 + (-4) = -2.$$

$$s(2) = -2$$

Problem 7: Cosine motion

$$\text{Displacement: } \int_0^\pi \cos t \, dt = \sin \pi - \sin 0 = 0.$$

$$\cos t \geq 0 \text{ on } [0, \frac{\pi}{2}] \text{ and } \leq 0 \text{ after. Distance: } \int_0^{\pi/2} \cos t \, dt - \int_{\pi/2}^\pi \cos t \, dt = 1 - (-1) = 2.$$

Displacement 0; distance 2

Problem 8: Area between a parabola and a line

Intersections: $x^2 = 3x + 4$ gives $(x-4)(x+1) = 0$, so $x = -1$ and $x = 4$. On $[-1, 4]$, the line is on top.

$$A = \int_{-1}^4 (3x+4-x^2) dx = \left[\frac{3x^2}{2} + 4x - \frac{x^3}{3} \right]_{-1}^4 = \frac{56}{3} - \left(-\frac{13}{6} \right) = \frac{125}{6}.$$

$$\frac{125}{6}$$

Problem 9: Area with two intersection points

Set $x+2=x^2$: $x^2-x-2=0$, so $x=-1$ or $x=2$. The line is on top.

$$A = \int_{-1}^2 (x+2-x^2) dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2 = \frac{10}{3} - \left(-\frac{7}{6} \right) = \frac{9}{2}.$$

$$\frac{9}{2}$$

Problem 10: Area between exponentials

For $x \geq 0$, $e^x \geq e^{-x}$. $A = \int_0^{\ln 3} (e^x - e^{-x}) dx = [e^x + e^{-x}]_0^{\ln 3}$.

At $\ln 3$: $3 + \frac{1}{3} = \frac{10}{3}$. At 0: 2. $A = \frac{10}{3} - 2 = \frac{4}{3}$.

$$\frac{4}{3}$$

Problem 11: Area using y

$y^2=2y$ gives $y=0$ and $y=2$. For $0 < y < 2$, $2y > y^2$, so $x=2y$ is the right curve.

$$A = \int_0^2 (2y - y^2) dy = \left[y^2 - \frac{y^3}{3} \right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}.$$

$$\frac{4}{3}$$

Problem 12: Three intersection points

$x^3=x$ gives $x=-1,0,1$. On $[-1,0]$, $x^3 \geq x$; on $[0,1]$, $x \geq x^3$.

$$\int_{-1}^0 (x^3 - x) dx = \left[\frac{x^4}{4} - \frac{x^2}{2} \right]_{-1}^0 = 0 - \left(\frac{1}{4} - \frac{1}{2} \right) = \frac{1}{4}.$$

$$\int_0^1 (x - x^3) dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Total area $\frac{1}{2}$

Problem 13: Sine and cosine area

On $(\frac{\pi}{4}, \frac{5\pi}{4})$, $\sin x \geq \cos x$. $A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx = [-\cos x - \sin x]_{\pi/4}^{5\pi/4}$.

At $\frac{5\pi}{4}$: $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$. At $\frac{\pi}{4}$: $-\sqrt{2}$. $A = \sqrt{2} - (-\sqrt{2}) = 2\sqrt{2}$.

$$2\sqrt{2} \approx 2.828$$

Problem 14: Square cross sections

The curves meet where $\sqrt{x} = \frac{x}{2}$, that is $x=0$ and $x=4$. The side of the square at x is $\sqrt{x} - \frac{x}{2}$.

$$V = \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx = \int_0^4 \left(x - x^{3/2} + \frac{x^2}{4}\right) dx = 8 - \frac{64}{5} + \frac{16}{3} = \frac{8}{15}.$$

$$\frac{8}{15}$$

Problem 15: Semicircle cross sections

Diameter $= \sqrt{x}$, radius $= \frac{\sqrt{x}}{2}$. Area $= \frac{\pi}{2} \cdot \frac{x}{4} = \frac{\pi x}{8}$.

$$V = \int_0^4 \frac{\pi x}{8} dx = \frac{\pi}{8} \cdot 8 = \pi.$$

$$\pi$$

Problem 16: Triangle cross sections

Side $s = 6x - x^2$. Area $= \frac{\sqrt{3}}{4} s^2$. Limits 0 to 6.

$$V = \frac{\sqrt{3}}{4} \int_0^6 (36x^2 - 12x^3 + x^4) dx = \frac{\sqrt{3}}{4} \left(2592 - 3888 + \frac{7776}{5} \right) = \frac{\sqrt{3}}{4} \cdot \frac{1296}{5} = \frac{324\sqrt{3}}{5}.$$

$$\frac{324\sqrt{3}}{5} \approx 112.2$$

Problem 17: Disc method about the x-axis

Radius $\sqrt{x}$. $V = \pi \int_0^4 x dx = \pi \cdot \frac{16}{2} = 8\pi$.

$$8\pi$$

Problem 18: Disc method about the y-axis

Write $x = \sqrt{y}$, so the radius is $\sqrt{y}$. $V = \pi \int_0^4 y dy = \pi \cdot 8 = 8\pi$.

$$8\pi$$

Problem 19: Washer about the x-axis

They intersect at $x=0$ and $x=1$; $y=x$ is outer. $V = \pi \int_0^1 (x^2 - x^6) dx = \pi \left(\frac{1}{3} - \frac{1}{7} \right) = \frac{4\pi}{21}$.

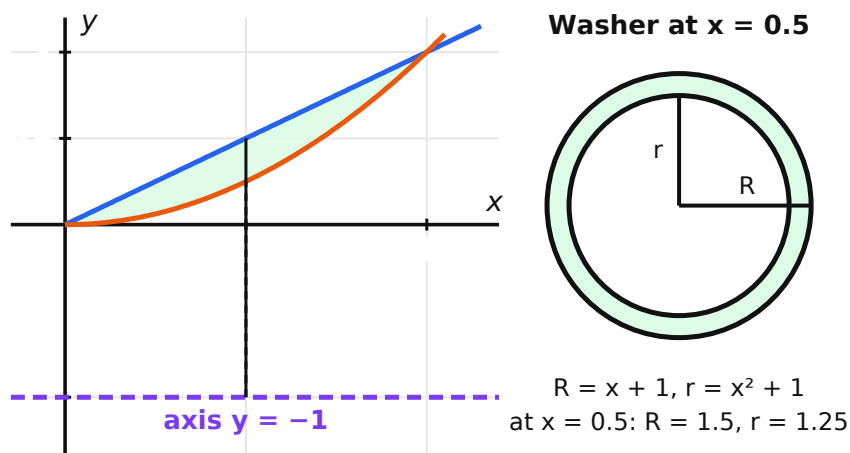
$$\frac{4\pi}{21}$$

Problem 20: Washer about a horizontal line

Distances from $y = -1$: outer $R = x+1$, inner $r = x^2+1$.

$$V = \pi \int_0^1 [(x+1)^2 - (x^2+1)^2] dx = \pi \int_0^1 (2x - x^2 - x^4) dx = \pi \left(1 - \frac{1}{3} - \frac{1}{5}\right) = \frac{7\pi}{15}.$$

$$\frac{7\pi}{15}$$



Revolving about $y = -1$: outer radius $x+1$, inner radius x^2+1 .

Problem 21: Rotation about a vertical line

Use y : the curve is $x = y^2$ for $0 \leq y \leq 2$. The radius from $x = 4$ to the curve is $4 - y^2$.

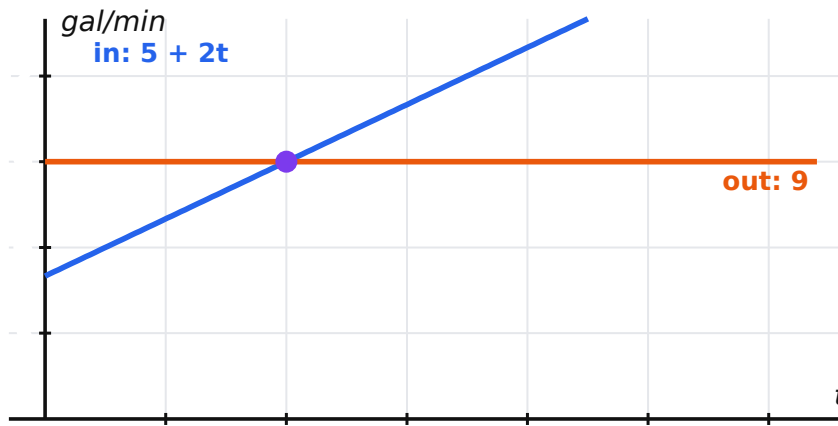
$$V = \pi \int_0^2 (4 - y^2)^2 dy = \pi \int_0^2 (16 - 8y^2 + y^4) dy = \pi \left(32 - \frac{64}{3} + \frac{32}{5}\right) = \frac{256\pi}{15}.$$

$$\frac{256\pi}{15}$$

Problem 22: Water in a tank

$$W'(t) = (5 + 2t) - 9 = 2t - 4. \quad W(6) = 100 + \int_0^6 (2t - 4) dt = 100 + (36 - 24) = 112.$$

$W' = 0$ at $t = 2$, and W' changes from negative to positive. $W(2) = 100 + (4 - 8) = 96$. Endpoints: $W(0) = 100$, $W(6) = 112$.



Inflow and outflow rates for the tank.

$W(6) = 112$ gallons; least at $t = 2$ minutes (96 gallons)

Problem 23: Oil leak

$$\int_0^5 12e^{-0.2t} dt = [-60e^{-0.2t}]_0^5 = 60(1 - e^{-1}) \approx 37.93.$$

About 37.93 gallons

Problem 24: Average temperature

$$\int_0^{12} 10 \sin \frac{\pi t}{12} dt = 10 \cdot \frac{12}{\pi} \left[-\cos \frac{\pi t}{12} \right]_0^{12} = \frac{120}{\pi} \cdot 2 = \frac{240}{\pi}.$$

$$\text{Average} = 60 + \frac{1}{12} \cdot \frac{240}{\pi} = 60 + \frac{20}{\pi} \approx 66.4.$$

About 66.4°F