

AP Calculus AB · Applications of Integration · Lesson

Common Core AP Calculus AB Unit 8 · mathovore.fr

Applications of integration use definite integrals to measure real quantities: average value, displacement and distance, area between curves, volumes from cross sections, volumes of revolution by discs and washers, and net change from rates in and out. Each one adds up thin slices.

AP Calculus ABCommon Core: AP Calculus AB Unit 8

Key facts:

- Average value: $\frac{1}{b-a} \int_a^b f(x) \, dx$.
- Area between curves: integrate top minus bottom, or right minus left.
- Volume with cross sections: $\int A(x) \, dx$; washers: $\pi \int (R^2 - r^2) \, dx$.
- Net change: $A(b) = A(a) + \int_a^b A'(t) \, dt$.

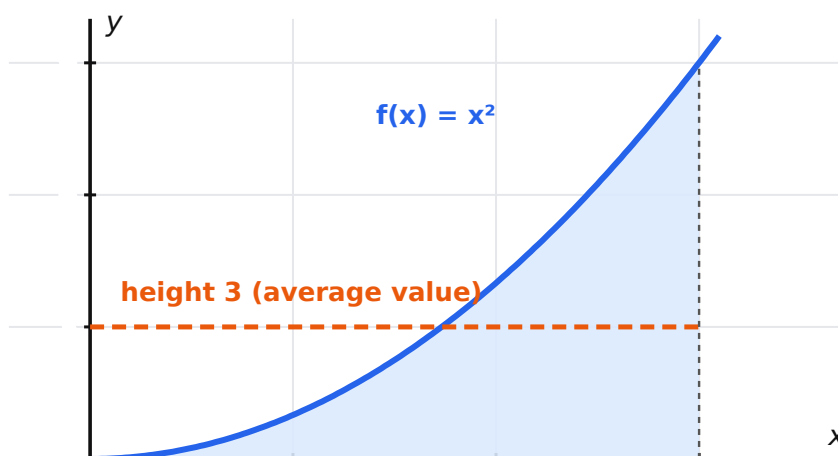
What is the average value of a function?

The **average value** of a continuous function f on $[a, b]$ is the height of the rectangle on $[a, b]$ that has the same area as the region under the graph.

Average value formula

$$f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx$$

The Mean Value Theorem for integrals says some c in $[a, b]$ has $f(c) = f_{\text{avg}}$.



The area under $f(x) = x^2$ on $[0, 3]$ equals a rectangle of height 3.

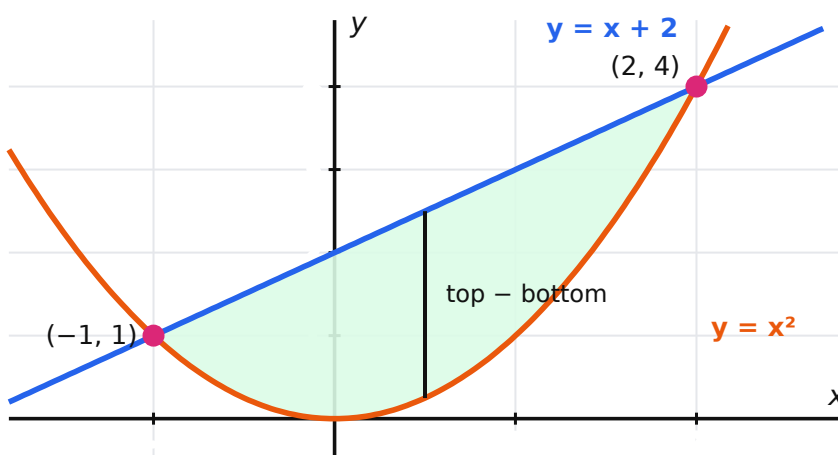
For $f(x) = x^2$ on $[0, 3]$, $\int_0^3 x^2 dx = 9$, so $f_{\text{avg}} = \frac{9}{3} = 3$.

How do integrals describe motion?

If $v(t)$ is velocity, then $\int_a^b v(t) dt$ is the **displacement** (net change in position) and $\int_a^b |v(t)| dt$ is the **total distance**. To find distance, split the interval wherever v changes sign. Position is $s(b) = s(a) + \int_a^b v(t) dt$.

How do you find the area between two curves?

Subtract the lower curve from the upper curve and integrate over the interval between intersections.

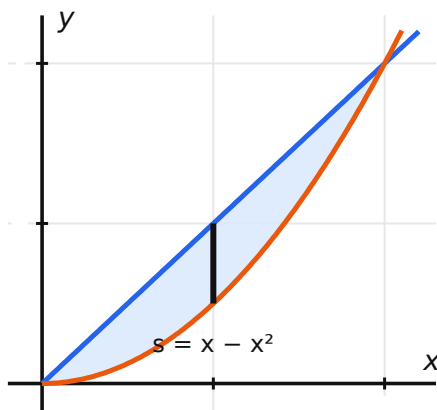


The region between $y = x + 2$ and $y = x^2$; each slice has length top minus bottom.

Here $A = \int_{-1}^2 [(x+2) - x^2] dx = \frac{9}{2}$. If the region is easier to describe with y , use $\int_c^d [x_{\text{right}} - x_{\text{left}}] dy$. When the curves cross more than twice, find every intersection and integrate each piece with its own top and bottom.

How do cross sections give volume?

Slice a solid perpendicular to an axis. If each slice has area $A(x)$, the volume is $V = \int_a^b A(x) dx$. Let s be the length of the slice in the base.



Cross section: square



$$\text{area} = s^2 = (x - x^2)^2$$

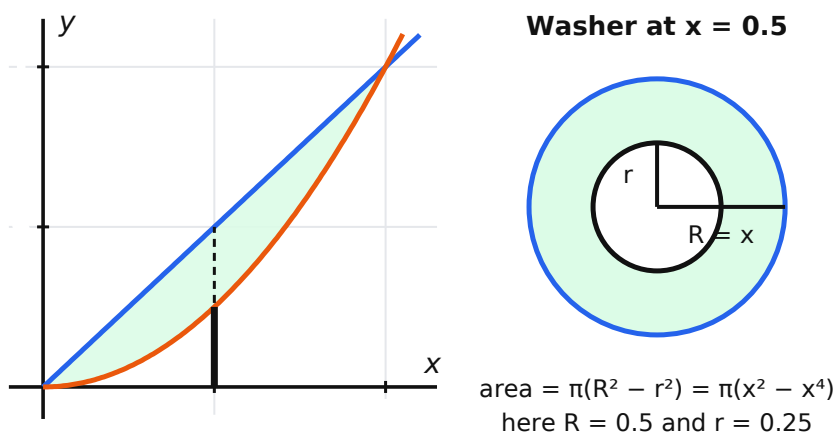
slice at $x = 0.5$: $s = 0.25$

A square cross section stands on each slice of the base region.

Cross section	Area in terms of s
Square	s^2
Semicircle (diameter s)	$\frac{\pi}{8}s^2$
Isosceles right triangle, hypotenuse s	$\frac{1}{4}s^2$
Equilateral triangle	$\frac{\sqrt{3}}{4}s^2$

How do discs and washers work?

Revolving a region about a line makes circular slices. With no hole use πR^2 ; with a hole use $\pi(R^2 - r^2)$, where R and r are measured from the axis of rotation.



Revolving a slice about the x -axis makes a washer: outer radius x , inner radius x^2 .

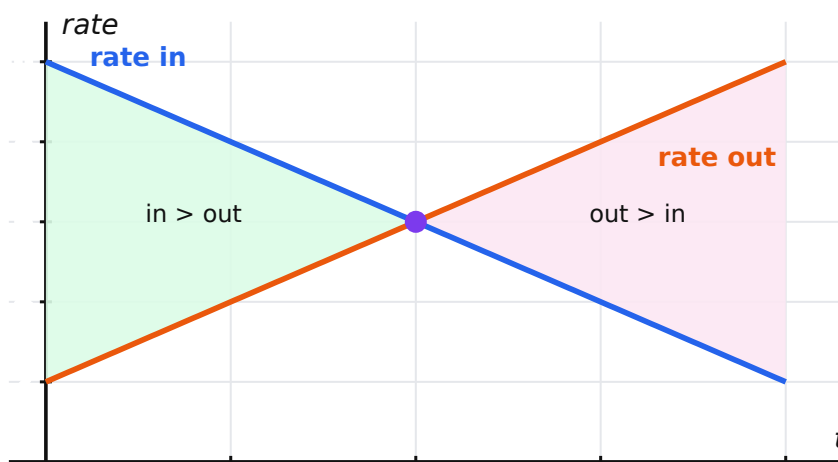
1. Sketch the region and the axis; draw one slice.
2. Write outer radius R and inner radius r as distances from the axis.
3. Integrate $\pi(R^2 - r^2)$ over the limits.

Example. Revolve the region between $y = x$ and $y = x^2$ about the x -axis:

$$V = \pi \int_0^1 (x^2 - x^4) dx = \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}.$$

How does accumulation work with rates in and out?

If $A'(t)$ is the rate in minus the rate out, then $A(b) = A(a) + \int_a^b A'(t) dt$. The amount is largest or smallest where the rates are equal and the sign of A' changes.



Amount increases while the rate in exceeds the rate out, and decreases afterward.

Applications of Integration: frequently asked questions

How do you find the average value of a function on an interval?

Integrate the function over $[a, b]$ and divide by the interval length $b - a$. For $f(x) = x^2$ on $[0, 3]$, the integral is 9, so the average value is 3.

What is the difference between displacement and total distance?

Displacement is $\int v(t) dt$, which lets backward motion cancel forward motion. Total distance is $\int |v(t)| dt$, found by splitting the interval where v changes sign.

When should you integrate with respect to y to find area?

Use y when the curves are written as x in terms of y , or when one dx integral would need several pieces but one dy integral needs only a right curve minus a left curve.

How do you find volume with known cross sections?

Write the area of one slice as a function of the position, using the slice length in the base as s . Then integrate that area between the limits of the base.

What is the difference between the disc and washer methods?

Discs have no hole, so each slice has area πR^2 . Washers have a hole, so each slice has area $\pi(R^2 - r^2)$. Measure both radii from the axis of rotation, even when it is not an axis.